

Warped Kaluza-Klein Towers Revisited

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Abstract

Inspired by the warped Randall Sundrum scenario proposed to solve the mass scale hierarchy problem with a compactified fifth extra dimension, a similar model with no metric singularities has been elaborated. In this framework, the Kaluza-Klein reduction equations for a real massless scalar field propagating in the bulk have been studied carefully from the point of view of hermiticity so as to formulate in a mathematically rigorous way all the possible boundary conditions and corresponding mass eigenvalue towers and tachyon states. The physical masses as observable in our four-dimensional brane are deduced from these mass eigenvalues depending on the location of the brane on the extra dimension axis. Examples of mass towers and tachyons and related field probability densities are presented from numerical computations performed for some arbitrary choices of the parameters of the model.

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1 Introduction

In a previous paper [1], we reanalysed mathematically, within the Arkhani-Ahmed, Dimopoulos, Dvali [2] large extra dimensions model, the procedure of generation of the Kaluza-Klein masses [3], stressing that it is the momentum squared in the extra dimensions (and not the momentum itself) which is the physically relevant quantity and hence corresponds to an operator which must essentially be hermitian. For illustration purpose, we restricted ourselves to the case of a five-dimensional massless real scalar field supposed to propagate in the flat five-dimensional bulk. The extra dimension is compactified to a finite range, say $[0, 2\pi R]$, either on a circle (then R is interpreted as the radius of the circle) or on a finite strip (then $2\pi R = L$ is the length of the strip). All the allowed boundary conditions resulting from the requirement that the extra dimension momentum squared must be a mathematically precisely-defined symmetric operator have been established (see also [4]). We deduced from them, besides the usual regularly spaced Kaluza-Klein mass towers, new towers with non regular mass spacing and tachyons. These considerations should be extended to vector and tensor fields.

In this article, inspired by the Randall and Sundrum scenario [5], we have developed a model based on a warped space with one extra dimension and basic parameters chosen so as to solve similarly the mass scale hierarchy problem.

In Randall-Sundrum, the fifth dimension s is compactified to an orbifold of radius R . A so called Planck brane is located at $s = 0$ while the TeV brane or Standard Model brane is at $s = \pi R$. We depart from the original scenario, postulating that the compactification is on a strip, that the metric has no singularity and that only one particular brane, the TeV brane on which we live as a four dimensional observer, has to be considered physically.

In this framework and again restricting to a real massless scalar field propagating in the bulk we have carefully studied the hermiticity properties of the operators in the Kaluza-Klein reduction equations for the adopted five dimensional warped space. We have enumerated all the allowed boundary conditions and from them we have deduced the corresponding Kaluza-Klein mass eigenstate towers and tachyon states and have studied their main properties.

As will be shown, the values of the observable physical masses in the Kaluza-Klein towers can be deduced from the mass eigenvalues and depend on the particular location of our four dimensional brane in s as do the field

probability densities which are related (when dynamics and kinematics are included) to the overall probability that the associated mass states would appear to the observer.

In the five dimensional bulk, we also postulate that all the dimensionfull parameters are scaled with a unique mass, the Planck mass M_{Pl} . It then happens that, by an adequate choice of the reduced parameters defining the model, all the low lying physical masses obtained from the eigenvalues of the Kaluza-Klein reduction equation are of order TeV for an observer living in our 4-dimensional brane.

2 Operators in the five-dimensional warped space. Mathematical considerations

The warped five dimensional space with coordinates $x^A, A = 0, 1, 2, 3, 5$ is composed of a flat $SO(1, 3)$ invariant infinite four-dimensional subspace labeled by x^μ ($\mu = 0, 1, 2, 3$) with signature $(+, -, -, -)$ and a spacelike fifth dimension with coordinate $x^5 \equiv s$ on the finite strip $0 \leq s \leq 2\pi R$. The metric, unique up to rescaling,

$$dS^2 = g_{AB} dx^A dx^B = e^{-2ks} dx_\mu dx^\mu - ds^2 \quad (1)$$

satisfies Einstein's equations with a stress-energy tensor identically zero and a bulk negative cosmological constant Λ as the unique origin of the induced Riemann metric (1). Indeed k , here chosen positive (see the discussion about the sign of k in Appendix(B)), and Λ are related by

$$k = \sqrt{-\frac{\Lambda}{6}} \ , \quad k > 0 \ . \quad (2)$$

A free massless scalar field $\Phi(x, s)$ in this warped space satisfies the invariant equation

$$\square_{\text{Riemann}} \Phi \equiv \frac{1}{\sqrt{g}} \partial_A \sqrt{g} g^{AB} \partial_B \Phi = 0 \ . \quad (3)$$

From the metric (1), $\sqrt{g} = e^{-4ks}$ and Eq.(3) becomes

$$\left(e^{2ks} \square_4 - e^{4ks} \partial_s e^{-4ks} \partial_s \right) \Phi(x^\mu, s) = 0 \quad (4)$$

where $\square_4 = \partial_\mu \partial^\mu$ is the usual four dimensional d'Alembertian operator .

We now carry on a careful study of the hermiticity properties of the operators appearing in (4). For scalar fields, the invariant scalar product is

$$(\Psi, \Phi) = \int_{-\infty}^{+\infty} d^4x \int_0^{2\pi R} ds \sqrt{g} \Psi^*(x, s) \Phi(x, s) . \quad (5)$$

Remember that an operator A with dense domain $D(A)$

- is symmetric for a scalar product if

$$(\Psi, A\Phi) = (A\Psi, \Phi) \quad (6)$$

for all the vectors $\Psi \in D(A)$ and $\Phi \in D(A)$, i.e. if the adjoint operator A^\dagger of the operator A is an extension of A : $A^\dagger\Phi = A\Phi$ for all $\Phi \in D(A)$ and $D(A^\dagger) \supset D(A)$,

- is self-adjoint if $A^\dagger\Phi = A\Phi$ for all $\Phi \in D(A)$ and moreover $D(A^\dagger) = D(A)$, i.e. if the operator is symmetric and if the equation (6) cannot be extended naturally to vectors Ψ outside $D(A)$.

We will call a differential operator which is symmetric up to boundary conditions a formally symmetric operator.

Symmetric Operators

For the scalar product (5), the operator \square_{Riemann} (3) is formally symmetric. The two operators which appear in (4) have the following properties

$$\begin{aligned} A_1 &\equiv e^{2ks} \square_4 && \text{is self-adjoint} \\ A_2 &\equiv e^{4ks} \partial_s e^{-4ks} \partial_s && \text{is formally symmetric .} \end{aligned} \quad (7)$$

By partial integration, one finds that the domain condition for the formally symmetric operator A_2 to be symmetric is

$$\left[\left(\Psi^* (\partial_s \Phi) - (\partial_s \Psi^*) \Phi \right) \right]_{2\pi R} = e^{8\pi k R} \left[\left(\Psi^* (\partial_s \Phi) - (\partial_s \Psi^*) \Phi \right) \right]_0 \quad (8)$$

which means that any field Φ and its derivative, both evaluated at $s = 0$ and $s = 2\pi R$, have to satisfy the same specific linear relations. These relations which express the boundary conditions will be studied carefully later.

Commuting operators

The two operators defined above (7) do not commute and hence cannot be diagonalized together.

Multiplying equation (4) on the left by e^{-2ks} leads to the following equivalent equation

$$\left(\square_4 - e^{2ks} \partial_s e^{-4ks} \partial_s \right) \Phi(x^\mu, s) = 0 \quad (9)$$

which defines two commuting operators with the following properties

$$\begin{aligned} B_1 &\equiv \square_4 && \text{is self-adjoint} \\ B_2 &\equiv e^{2ks} \partial_s e^{-4ks} \partial_s && \text{is not even formally symmetric .} \end{aligned} \quad (10)$$

Puzzle

So we are facing a puzzle

- either the two operators A_1, A_2 (7) are formally symmetric but do not commute,
- or the two operators B_1, B_2 (10) commute but the second is not formally symmetric.

Solving the puzzle

The puzzle can be solved remembering recent discussions about non-hermitian operators having real eigenvalues [6]. It was shown in [7] that in many cases these non hermitian operators are in fact equivalent to hermitian operators by a non-unitary change of basis. This is the case here. Indeed after some algebra, considering the non-unitary transformation induced by $V(s) = e^{ks}$, one finds

$$\begin{aligned} \tilde{B}_1 &= V B_1 V^{-1} = B_1 \\ \tilde{B}_2 &= V B_2 V^{-1} \\ \tilde{\Phi} &= V \Phi \end{aligned} \quad (11)$$

with the result that \tilde{B}_1 and \tilde{B}_2 are at least formally symmetric operators for the induced scalar product

$$(\tilde{\Psi}, \tilde{\Phi}) = \int_{-\infty}^{+\infty} d^4x \int_0^{2\pi R} ds e^{-6ks} \tilde{\Psi}^*(x, s) \tilde{\Phi}(x, s) . \quad (12)$$

It is also easy to see that the natural conditions for the operator \widetilde{B}_2 to be symmetric when acting on the space of vectors $\widetilde{\Phi}$ are

$$\left[\left(\widetilde{\Psi}^* (\partial_s \widetilde{\Phi}) - (\partial_s \widetilde{\Psi}^*) \widetilde{\Phi} \right) \right]_{2\pi R} = e^{12\pi k R} \left[\left(\widetilde{\Psi}^* (\partial_s \widetilde{\Phi}) - (\partial_s \widetilde{\Psi}^*) \widetilde{\Phi} \right) \right]_0 . \quad (13)$$

These conditions turn out to be fully compatible with those (8) obtained for $\Phi(x^\mu, s)$ from the requirement that A_2 is symmetric.

Thus even though the operator B_2 is not symmetric, it is equivalent through a change of basis to a symmetric operator \widetilde{B}_2 and hence will produce real eigenvalues which will be related to the Kaluza-Klein tower masses as will appear below.

3 The Kaluza-Klein reduction equations

To simplify the discussion, we concentrate on the particular case of a real massless scalar field. The general procedure to solve the basic equation (9) along the Kaluza-Klein reduction method is well-known. One supposes that the field $\Phi(x^\mu, s)$ is a linear combination of terms where the variables x^μ and s separate

$$\Phi(x^\mu, s) = \sum_n \phi_n^{[x]}(x^\mu) \phi_n^{[s]}(s) . \quad (14)$$

Then $\Phi(x^\mu, s)$ is a solution of (9) if

$$B_1 \phi_n^{[x]}(x^\mu) \equiv \square_4 \phi_n^{[x]}(x^\mu) = -m_n^2 \phi_n^{[x]}(x^\mu) \quad (15)$$

$$B_2 \phi_n^{[s]}(s) \equiv e^{2ks} \partial_s e^{-4ks} \partial_s \phi_n^{[s]}(s) = -m_n^2 \phi_n^{[s]}(s) . \quad (16)$$

From the arguments given above one then concludes

1. The operators B_1 and B_2 commute and can indeed be diagonalized simultaneously
2. The operator B_2 is equivalent to a formally symmetric operator through a non-unitary change of basis and hence, taking into account boundary conditions compatible with (8), Eq.(16) gets real m_n^2 eigenvalues which can be positive, zero or negative.

3. The operator B_1 is self-adjoint. By (15), the solutions for $m_n^2 > 0$ correspond to four-dimensional physical particles, those with $m_n^2 = 0$ to four-dimensional massless particles and those with $m_n^2 < 0$ to four-dimensional tachyons.

The boundary restrictions (8) can be conveniently rewritten

$$\begin{aligned} & \left[\left(\psi_p^{[s]} (\partial_s \phi_n^{[s]}) - (\partial_s \psi_p^{[s]}) \phi_n^{[s]} \right) \right]_{2\pi R} \\ &= \left[\left((e^{4\pi k R} \psi_p^{[s]}) (\partial_s e^{4\pi k R} \phi_n^{[s]}) - (\partial_s e^{4\pi k R} \psi_p^{[s]}) (e^{4\pi k R} \phi_n^{[s]}) \right) \right]_0. \end{aligned} \quad (17)$$

When $k = 0$, these restrictions are identical to the restrictions applicable in the fully flat case which we studied in [1]. As a consequence, the boundary conditions compatible with (17) can be obtained by simply replacing $\phi_n^{[s]}(0)$ and $\partial_s \phi_n^{[s]}(0)$ by respectively $e^{4\pi k R} \phi_n^{[s]}(0)$ and $e^{4\pi k R} \partial_s \phi_n^{[s]}(0)$ in the boundary conditions which we listed for the flat case.

In fact, each field must satisfy the same set of boundary conditions which consist of at least two linear relations. This set defines specific domains in the Hilbert space. In Table(1), we give all the possible independent sets of boundary conditions expressed by just two linear relations. The box condition appears in Case A6.

Boundary conditions expressed with more than two linear relations can be considered as restrictions applied to the cases with two relations. The domain of the operator is then reduced. For three linear relations, the sets are given in Table(2). The set involving four relations consists of $\phi_n^{[s]}(0) = 0$, $\phi_n^{[s]}(2\pi R) = 0$, $\partial_s \phi_n^{[s]}(0) = 0$, $\partial_s \phi_n^{[s]}(2\pi R) = 0$, which can only be satisfied by the trivial field $\phi_n^{[s]}(s) = 0$ and can thus be forgotten.

The boundary conditions have to be imposed to the general solutions of the equation (16) which are

- For $m_n^2 > 0$, the solutions are linear superpositions of the Bessel functions J_2 and Y_2 (see Appendix (A))

$$\phi_n^{[s]}(s) = e^{2ks} \left(\sigma_n J_2 \left(\frac{m_n e^{ks}}{k} \right) + \tau_n Y_2 \left(\frac{m_n e^{ks}}{k} \right) \right). \quad (18)$$

- The solution for $m_0^2 = 0$ is

$$\phi_0^{[s]}(s) = \sigma_0 e^{4ks} + \tau_0 . \quad (19)$$

- The tachyon solutions for $m_t^2 = -h_t^2 < 0$ are linear superpositions of the modified Bessel functions I_2 and K_2 (see Appendix (A))

$$\phi_t^{[s]}(s) = e^{2ks} \left(\sigma_t I_2 \left(\frac{h_t e^{ks}}{k} \right) + \tau_t K_2 \left(\frac{h_t e^{ks}}{k} \right) \right) . \quad (20)$$

- In the above formulae, $\sigma_n, \tau_n, \sigma_0, \tau_0, \sigma_t$ and τ_t are constants.

4 Physical considerations. Masses

We now extend our considerations to potentially physical consequences including for example the possible discovery of TeV warped states at high energy colliders. In the next subsections, we first discuss the magnitude of the parameters k and R which occur in the model and also the magnitude of the parameters $\alpha_i \dots$ which define the boundary conditions, postulating that there is only one scale in the model, the Planck mass. We then write explicitly the equations determining the mass eigenvalues. We discuss the interpretation of these mass eigenvalues in terms of the physical masses as they would be observed in our brane. In particular the conditions for the existence of zero mass states and of tachyons mass states are deduced. Examples are finally given and discussed.

4.1 The parameters

The general philosophy underlying the warped approach, which was proposed to solve the hierarchy problem, is that there is only one mass scale, the Planck mass $M_{Pl} \approx 1.22 \cdot 10^{16}$ TeV, and hence that any dimensionfull parameter p with energy dimension d is of order M_{Pl}^d . More precisely

$$p = \bar{p} (M_{Pl})^d \quad (21)$$

with \bar{p} a pure number of order one. In particular, $k = \bar{k} M_{Pl}$ and $R = \bar{R} M_{Pl}^{-1}$.

As stated above (2), the parameter k is chosen positive. As will be seen hereafter, it appears that for an observer sitting at $s = 0$ a reasonable choice for the product kR is

$$kR = \bar{k}\bar{R} \approx 6.3 \quad (22)$$

as the resulting lowest masses in the Kaluza-Klein towers would then be of order 1 TeV.

4.2 The solutions

We restrict ourselves to all the boundary conditions which are expressed by two relations (see Table(1)). We first discuss the Kaluza-Klein mass towers in general, then in particular the towers which have a mass zero as their lowest mass state and finally the towers with a tachyon state.

Towers $m_n^2 > 0$

Two boundary relations being applied to the fields (18) lead to two linear homogeneous equations in terms of the parameters σ_n and τ_n . The coefficients turn out to be linear combinations of Bessel functions evaluated for $s = 0$ and $s = 2\pi R$, i.e. with arguments respectively equal to

$$\begin{aligned} F_0 &= \frac{m_n}{k} \\ F_2 &= e^{2\pi kR} \frac{m_n}{k} . \end{aligned} \quad (23)$$

In order to find non trivial solutions for σ_n and τ_n and hence non trivial fields $\phi_n^{[s]}$, the relevant determinant has to be equal to zero. This leads to the mass equation whose solutions provide for chosen parameters fixing the boundary conditions the mass eigenvalues m_n building up the related tower. Once these mass eigenvalues are known, the corresponding σ_n/τ_n ratios are deduced from one of the two original equations. For each set of allowed boundary conditions, both the mass equation and a σ_n/τ_n relation are given in Table(4).

Zero mass states

With two boundary relations being applied to the fields (19), one obtains again two linear homogeneous equations in terms of the parameters σ_0 and τ_0 . The condition that the relevant determinant is zero is in fact the constraint

that has to be satisfied by the boundary condition parameters for a zero mass state to exist. One of the two equations fixes again the ratio σ_0/τ_0 .

For each of the allowed set of boundary conditions, both the parameter constraint and a σ_0/τ_0 equation are given in Table(3). Since $e^{4\pi k R}$ takes such a large value and since the reduced parameters $\bar{\alpha}_i, \dots$ are assumed to be of order one, approximate relations, valid to a high degree of precision, are easily obtained. They are also listed in Table(3). We should remark that there is no zero mass state for the box boundary condition (Case A6).

The parameter constraint equation for a zero mass state defines a surface in the parameter space (see approximate equation in Table(3)). In many cases, if one follows a path in the parameter space which crosses the parameter constraint surface, the lowest mass eigenvalue goes smoothly toward zero, takes the value zero as the path goes through the surface and emerges as a tachyon state with low $h^2 = -m^2$. Examples will be given in subsection(4.4) for the boundary Cases A1, A4 and A3. A different behavior shows up in the Case A3 when in particular $\bar{\rho}_1$ passes zero. The zero mass appears suddenly as the surface is crossed. No small mass, no tachyons appear on either sides of the surface. This results from the fact that the solution $m^2 = 0$ is of higher order.

Tachyon states $h^2 = -m^2 > 0$

The equations for h^2 can simply be obtained from the equations giving the tower masses by replacing the Bessel functions J_n by I_n and Y_n by $(-1)^n K_n$. They are summarized in Table(5) for each set of boundary conditions.

4.3 Physical interpretation of the mass eigenvalues

At this stage of the discussion, in order to make connection with physics, one has to take into account the position $s = s_0$ ($0 \leq s_0 \leq 2\pi R$) where the TeV brane, the brane in which we live, is located. Indeed the deduction of the physical masses in terms of the mass eigenvalues depends crucially on this position.

In our brane, the space time part of the metric (1) has a factor e^{-2ks_0} . By a change of variable

$$\tilde{x}_\mu = e^{-ks_0} x_\mu \quad (24)$$

the space time metric in normal local units becomes $dS^2 = d\tilde{x}_\mu d\tilde{x}^\mu$. Hence

the physical mass m_{n,s_0} obeys the equation

$$\tilde{\square}_4 \tilde{\phi}_n^{[x]}(\tilde{x}^\mu) = -m_{n,s_0}^2 \tilde{\phi}_n^{[x]}(\tilde{x}^\mu) . \quad (25)$$

Comparing (25) and (15) we find

$$m_{n,s_0} = e^{ks_0} m_n . \quad (26)$$

This gives the relation between the m_n eigenvalues which appear in Eqs.(15) and (16) and the observable physical masses m_{n,s_0} in the brane. In the case $s_0 = 0$, the eigenvalues and the physical masses are equal ($m_{n,0} = m_n$). The low lying masses are of order 1 TeV when $kR = 6.3$. If s_0 differs appreciably from zero one sees that m_{n,s_0} may become large enough to spoil the hierarchy solution. However, as will be seen later, the solution of the problem is fully restored by an adequate increase of kR

4.4 Examples

In this subsection, we show for illustration the results of some numerical computations for Kaluza-Klein mass eigenvalue towers and field probability densities corresponding to two sets of boundary conditions belonging to Case A4 and Case A1 respectively. We also make at the end some introductory comments about the mass tower structure in the Case A3

In all the examples, the parameter kR has been set equal to value 6.3 and the parameter \bar{k} has been chosen equal to one for convenience, so

$$kR = 6.3 \quad , \quad \bar{k} = 1 . \quad (27)$$

The extension to other values of kR and of \bar{k} is outlined.

As stated in (4.3), the computed mass eigenvalues would be the physical masses for a four-dimensional observer at $s_0 = 0$. The physical masses for $s_0 \neq 0$ can be deduced from (26).

Case A4 mass eigenvalues

The case A4 is simpler since there is only one free parameter fixing the boundary condition. We remark that the mass equation for the tower eigenvalues (see Table(4)) is invariant under the following rescaling with the arbitrary

parameter λ

$$\begin{aligned}
\bar{R} &\rightarrow \frac{\bar{R}}{\lambda} \\
\bar{k} &\rightarrow \lambda \bar{k} \\
\bar{\kappa} &\rightarrow \lambda \bar{\kappa} \\
m &\rightarrow \lambda m
\end{aligned} \tag{28}$$

which leaves $kR = \bar{k}\bar{R}$ invariant. Hence

$$\lambda m_i(\bar{k}, \bar{\kappa}) = m_i(\lambda \bar{k}, \lambda \bar{\kappa}) \tag{29}$$

which allows one to determine the tower mass states for other values of \bar{k} from the mass states corresponding to our choice $\bar{k} = 1$. Indeed

$$m_i(\bar{k}, \bar{\kappa}) = \bar{k} m_i\left(1, \frac{\bar{\kappa}}{\bar{k}}\right) . \tag{30}$$

The ten first mass eigenvalues m_i ($i = 1, \dots, 10$) in the towers are given in the Table(6) (remember (27)) and for a few values of the boundary parameter $\bar{\kappa}$ distributed around $\bar{\kappa} = 4$ for which there is a zero mass state (see Table(3)).

A few comments are worth making

- Referring to the Table, one sees that the average distance $\Delta m_i = m_{i+1} - m_i$ with $i \geq 2$ between two consecutive states decreases very slowly along a tower and is of the order,

$$\Delta m_i \approx 0.25 \text{ TeV} . \tag{31}$$

- As a result of our numerical computations ($\bar{k} = 1$) performed for neighbouring values of $\bar{k}\bar{R}$, namely

$$\bar{k}\bar{R} = 6.3 + \Delta(\bar{k}\bar{R}) , \tag{32}$$

we have found that the corresponding mass eigenvalues in a given tower can be deduced precisely by

$$m_i^{[\bar{k}\bar{R}]} = m_i^{[6.3]} e^{-2\pi\Delta(\bar{k}\bar{R})} \quad \text{at fixed } \bar{k} . \tag{33}$$

- The physical masses of the tower in the $s = s_0$ brane, i.e. $m_{i,s_0}^{[\bar{k}\bar{R}]}$, are obtained from (26) (taking into account (27)) by

$$m_{i,s_0}^{[6.3]} = m_{i,0}^{[6.3]} e^{\bar{k}\bar{s}_0} \quad (34)$$

and hence, more generally

$$m_{i,s_0}^{[\bar{k}\bar{R}]} = m_{i,0}^{[6.3]} e^{\bar{k}\bar{s}_0 - 2\pi\Delta(\bar{k}\bar{R})} . \quad (35)$$

This is true at \bar{k} fixed. Remember that the consequences of a change of \bar{k} (at fixed $\bar{k}\bar{R}$) can be obtained from the above rescaling properties (30).

- The differences between the masses in the towers are exponentially sensitive to the parameter $\bar{k}(\bar{s}_0 - 2\pi\Delta(\bar{R}))$ while it is multiplicatively sensitive to \bar{k} . Should a Kaluza-Klein tower be discovered an approximate value of $\bar{k}(\bar{s}_0 - 2\pi\Delta(\bar{R}))$ could be deduced.
- A particular attention has to be drawn on the first state. Following the path in the $\bar{\kappa}$ parameter space, going up from a large negative value of $\bar{\kappa}$, one sees that the first mass eigenvalue m_1 decreases faster than m_2 and gets equal to zero as one reaches the surface $\bar{\kappa} = 4$ ($\bar{k} = 1$) of the zero mass constraint (see Table(3)). At that point, $m_2 - m_1$ is about twice the average Δm_i value. Once the zero mass surface is passed, the first mass eigenvalue disappears and a tachyon state develops with h increasing rapidly.
- The mass m_i increases rather slowly when $\bar{\kappa}$ decreases toward $-\infty$ and becomes equal to the mass m_{i+1} corresponding to $\bar{\kappa} = +\infty$, exhibiting continuity of the masses as functions of $\bar{\kappa}$.

Case A1 mass eigenvalues

For the Case A1, the mass equation (see Table (4)) is invariant under the rescaling analogous to (28), namely $[\bar{R} \rightarrow \bar{R}/\lambda, \bar{k} \rightarrow \lambda\bar{k}, \bar{\alpha}_1 \rightarrow \bar{\alpha}_1, \bar{\alpha}_2 \rightarrow \bar{\alpha}_2/\lambda, \bar{\alpha}_4 \rightarrow \bar{\alpha}_4, m \rightarrow \lambda m]$, and hence

$$m_i(\bar{k}, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_4) = \frac{1}{\lambda} m_i\left(\lambda\bar{k}, \bar{\alpha}_1, \frac{\bar{\alpha}_2}{\lambda}, \bar{\alpha}_4\right) \quad (36)$$

$$m_i(\bar{k}, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_4) = \bar{k} m_i\left(1, \bar{\alpha}_1, \bar{k}\bar{\alpha}_2, \bar{\alpha}_4\right) . \quad (37)$$

The ten lowest mass eigenvalues in the Kaluza-Klein towers are given in Table(7) for an arbitrary choice of the boundary parameters $\bar{\alpha}_1 = 0.7$, $\bar{\alpha}_4 = 6.6286$ and for a set of values of $\bar{\alpha}_2$ including the value $\bar{\alpha}_2 = (\bar{\alpha}_1\bar{\alpha}_4 - 1)/(4\bar{\alpha}_1\bar{k}) = 1.3$ where the path in parameter space crosses the zero mass surface. The value $\bar{\alpha}_2 = 0$ is not only excluded but appears as a singular point. It is then convenient to vary the values of $\bar{\alpha}_2$ from zero to $+\infty$ and then from $-\infty$ back to zero. It should be noted that the passage through the value $\bar{\alpha}_2 = \pm\infty$ is smooth.

Apart from a few small differences, the main structure of the mass towers is essentially the same as for the Case A4 above. The mass eigenvalues and the physical masses corresponding to other values of \bar{R} and \bar{s}_0 (for \bar{k} fixed) can again be deduced by the same formulae (33)-(35). A change of \bar{k} follows from the rescaling equation (30).

Moreover, comparing the towers which have a zero mass state at their bottom (line $\bar{\kappa} = 4$ for Case A4 in Table(6) and line $\bar{\alpha}_2 = 1.3$ for Case A1 in Table(7)), we note that all the masses in these two towers are practically identical. This holds for any kR in the physically allowed range.

General comments about Case A3 mass eigenvalues

Summarizing

- The rescaling $[\bar{R} \rightarrow \bar{R}/\lambda, \bar{k} \rightarrow \lambda\bar{k}, \bar{\rho}_1 \rightarrow \lambda\bar{\rho}_1, \bar{\rho}_2 \rightarrow \lambda\bar{\rho}_2, m \rightarrow \lambda m]$, leads to the formulae

$$m_i(\bar{k}, \bar{\rho}_1, \bar{\rho}_2) = \frac{1}{\lambda} m_i(\lambda\bar{k}, \lambda\bar{\rho}_1, \lambda\bar{\rho}_2) \quad (38)$$

$$m_i(\bar{k}, \bar{\rho}_1, \bar{\rho}_2) = \bar{k} m_i\left(1, \frac{\bar{\rho}_1}{\bar{k}}, \frac{\bar{\rho}_2}{\bar{k}}\right) . \quad (39)$$

- The approximate parameter condition for the existence of a zero mass is (see Table(3))

$$\bar{\rho}_1(\bar{\rho}_2 - 4\bar{k}) = 0 . \quad (40)$$

For each of the two solutions $\bar{\rho}_1 = 0$ or $\bar{\rho}_2 = 4\bar{k}$, there indeed exists a zero mass state.

- For $\bar{\rho}_2 = 4$ (Table(8)) the mass tower is identical to the mass towers with a zero mass state in the cases A1 and A4 (Tables(7) and (6)), and this independently of the value of $\bar{\rho}_1$.

- For $\bar{\rho}_1 = 0$, besides the zero mass the other masses depend on the value of $\bar{\rho}_2$. When $\bar{\rho}_2$ moves toward 4, the lowest non zero mass in the tower converges also to zero.
- When $\bar{\rho}_2$ is fixed to a given value, all the towers corresponding to any value of $\bar{\rho}_1$ are identical, including the tachyon if it exists (i.e. for $\bar{\rho}_2 > 4$). There is of course an extra mass zero for $\bar{\rho}_1 = 0$. However, for $\bar{\rho}_1$ close to zero, neither a small mass particle nor a small h tachyon appears.

5 Physical Considerations. Probability densities

In the context of a given boundary case, once all the parameters are fixed and the mass eigenvalue tower is determined, there exists a unique field $\phi_n^{[s]}(s)$ for each mass eigenvalue leading to a normalized probability density field distribution $D_n(s)$ along the fifth dimension (5)

$$D_n(s) = \frac{\sqrt{g}(\phi_n^{[s]}(s))^2}{\int_0^{2\pi R} ds \sqrt{g}(\phi_n^{[s]}(s))^2} . \quad (41)$$

It is convenient to parametrize the s range $[0, 2\pi R]$ by the reduced variable x defined by

$$x = \frac{s}{2\pi R} \quad (42)$$

with range $[0, 1]$.

Case A1 field probability densities

We consider the tower labeled by $\bar{\alpha}_2 = 1.3$ in Table(7). The logarithm of the normalized probability density for the three mass eigenvalues $m_1 = 0$, $m_2 = 0.501$ TeV and $m_5 = 1.275$ TeV are given in Figure(1), (2) and (3) respectively as functions of x . The general trends are as follows:

- The probability density is a fast varying function of x . In a large part of the domain the logarithm increases or decreases linearly.
- A general pattern emerges. For m_i with i even, the probability density presents around $x = 0.5$ a very steep dip down to zero as a result of a brutal but continuous change of sign of the field at that point.

- Moreover, the mass m_i presents $i-1$ probability dips in the high x region $0.95 \leq x \leq 1$.

Relative mass eigenstate probabilities for given x_0

In a brane supposed to be located at a certain fixed x_0 ($s_0 = 2\pi R x_0$), it is directly possible to compare the probabilities of the different mass eigenstates in a given tower. Neglecting dynamical and kinematical effects related to the production in the available phase space, these probabilities given in Table(9) would account for the rate of appearance of the mass eigenvalue states to an observer sitting at this x_0 . Note however that the physical masses as seen by this observer (at $x_0 \neq 0$) are not the eigenvalue masses but vary with the s_0 in agreement with (35).

The ratios of probabilities densities are given in Table(9) for the first ten mass eigenstates and selected values of x in its range $[0, 1]$, arbitrarily normalizing to one the highest probability among the ten first masses considered.

- For x values outside the region where dips in the probability density appear i.e. in the two regions $0 \leq x < 0.40$ and $0.55 < x < 0.90$, the relative probabilities are very weakly dependent on x . In the first region $0 \leq x < 0.40$, the probability for m_1 dominates whereas in the second region $0.55 < x < 0.95$, it dominates for m_{10} .
- In the dip regions $0.45 \leq x \leq 0.5$ and $0.90 \leq x \leq 1$, a chaotic behavior shows up. Small variations of x may imply large fluctuations of the probabilities.

Relative probabilities in a given physical mass tower as a function of x_0

In Table(10), we have taken into account the increase of \bar{R} dictated by (35) ($\bar{R} = 6.3/(1 - x_0)$) as requested to keep the low lying physical masses exactly unchanged as one increases x_0 starting from $x_0 = 0$ where mass eigenvalues and physical masses coincide. We have limited ourselves to $\bar{R} \leq 50$. Indeed, larger values of \bar{R} obviously spoil the underlying philosophy that all the reduced parameters have to be of order one. The similarity between the results presented in the two Tables (9) and (10) should be noted though the physical interpretation is widely different.

Case A4 field probability densities

Summarizing

- Figures(4) and (5) show the logarithm of the normalized probability density as a function of x for $\bar{\kappa} = 4$, respectively for two mass eigenvalues chosen for illustration $m_1 = 0$ and $m_3 = 0.766$ TeV.
- All the probabilities are seen to increase very fast with x since the logarithm is essentially a linear function for $0.1 \leq x \leq 0.9$.
- Due to the boundary condition $\phi(0) = 0$ (Table(1)) there is a sharp dip for $s = 0$.
- In the high x region ($0.9 \leq x \leq 1$) the probability for the mass m_i exhibits $i-1$ sharp dips corresponding to zeros in the field.
- In the whole region $0.1 \leq x \leq 0.9$ the relative probabilities in a tower are very close to those of Case A1 for x between 0.5 and 0.9 (Table (9)).
- In the two extreme regions (where there are dips) the relative probabilities exhibit a chaotic behavior as in the dip regions for A1.
- The same considerations hold as for the case A1 regarding the requested readjustment of \bar{R} to keep the physical masses unchanged when the position x_0 is changed.

6 Conclusions

Inspired by the warped five-dimensional scenario of Randall and Sundrum and restricting to the case of a real massless scalar field supposed to propagate in the bulk, we have developed a similar warped model, keeping all the basic parameters adjusted in terms of the Planck mass as the only dimensionful scale. This, in the end, solves the mass scale hierarchy problem.

We have concentrated primarily on a careful study of the hermiticity (symmetry, self-adjointness) and commutativity of the operators susceptible to be used to validly establish the Kaluza-Klein reduction equations in the five dimensional warped space. Postulating that the fifth extra dimension s is compactified on a strip $0 \leq s \leq 2\pi R$ and that the metric has no discontinuity,

we have enumerated all the allowed boundary conditions. From them we have deduced all the Kaluza-Klein towers mass equations providing the mass eigenvalues, as well as the tachyon mass equations.

We have discussed how these mass eigenstates show up with physical masses depending on the location of our brane on the s axis.

As an illustration, we have carried on some numerical computations for the three sets of boundary conditions A1, A4 and A3 in order to visualize the structure of the towers and to investigate their main properties. The other cases can be studied along the same lines.

The structure of the eigenvalue towers depends generally in a sensitive way on the value of the basic parameter kR and to a smaller extend on the boundary parameters and on the reduced parameter \bar{k} . Apart from small differences, the main structure is the same in the three Cases considered.

One notices that the first tower masses are of the order TeV for kR around 6.3 as expected for solving the mass hierarchy problem.

In general, for specific values of the boundary parameters there exists a zero mass state. For parameters close to these values the first mass state in the tower is often either a particle with a small mass or a tachyon.

One observes that the mass spacing (discarding the first mass state) is very stable within a given tower and is exponentially sensitive to the value of $k(s_0 - 2\pi R)$ where s_0 is the position of our brane. Hence should a Kaluza-Klein tower be observed experimentally, a good estimation of this basic parameter would result.

The normalized field probability density for any physical mass state in a tower can easily be computed for fixed values of the boundary parameters as a function of \bar{k} , \bar{R} and x_0 . Neglecting dynamical and kinematical effects the ratios of the probabilities among the first masses in a given tower evaluated at x_0 would express the relative intensities of the eventually observed mass peaks.

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A Appendix: The Bessel Functions. Notations

The Bessel function J_p , as well as Y_p , satisfies

$$\begin{aligned} \left(y^2 \partial_y^2 + y \partial_y + (y^2 - p^2) \right) J_p(y) &= 0 \\ \partial_y J_p(y) + p \frac{J_p(y)}{y} - J_{p-1}(y) &= 0 \\ J_p(y) - 2(p-1) \frac{J_{p-1}(y)}{y} + J_{p-2}(y) &= 0 . \end{aligned} \quad (43)$$

A useful identity is

$$J_2(y)Y_1(y) - J_1(y)Y_2(y) - \frac{2}{\pi y} = 0 . \quad (44)$$

The modified Bessel function I_p , as well as K_p , satisfies

$$\left(y^2 \partial_y^2 + y \partial_y - (y^2 + p^2) \right) I_p(y) = 0 \quad (45)$$

and (note the sign differences)

$$\begin{aligned} \partial_y I_p(y) + p \frac{I_p(y)}{y} - I_{p-1}(y) &= 0 \\ \partial_y K_p(y) + p \frac{K_p(y)}{y} + K_{p-1}(y) &= 0 \\ I_p(y) + 2(p-1) \frac{I_{p-1}(y)}{y} - I_{p-2}(y) &= 0 \\ K_p(y) - 2(p-1) \frac{K_{p-1}(y)}{y} - K_{p-2}(y) &= 0 . \end{aligned} \quad (46)$$

The corresponding useful identity is

$$I_2(y)K_1(y) + I_1(y)K_2(y) - \frac{1}{y} = 0 . \quad (47)$$

B Appendix: Singularities

In the main part of the paper, we have analyzed the case of a warped space (1) induced by a cosmological constant (2) with a fifth dimension compactified to a strip $0 \leq s \leq 2\pi R$. We made the choice of a positive k everywhere in the space and postulated that there was no singularity.

First, the k negative case is simply related to the positive case by the exchange $s \leftrightarrow -s + 2\pi R$ which hence makes the sign of k an arbitrary choice.

However, it may be assumed that in some region of s the constant k is positive and in another region it is negative. With either k positive or negative there must be at least a singular point s_s where a transition in the metric occurs. Writing

$$\begin{aligned} \text{for } s < s_s & \quad dS^2 = Ce^{2k(s-s_s)} dx_\mu dx^\mu - ds^2 \\ \text{for } s = s_s & \quad dS^2 = C \quad dx_\mu dx^\mu - ds^2 \\ \text{for } s > s_s & \quad dS^2 = Ce^{-2k(s-s_s)} dx_\mu dx^\mu - ds^2 \end{aligned} \quad (48)$$

the metric is continuous at $s = s_s$ as it should but its first derivative has a discontinuity $\pm 4kC$ and its second derivative a δ -function behavior. In principle, there could be any finite number of such singularities.

If there is an even number of singularities ($2m, m \geq 1$), the strip compactification can be transformed to an orbifold compactification by identifying the edges 0 and $2\pi R$ closing the strip to a circle, allowing periodic or antiperiodic conditions. Without loss of generality, one may choose these $2m$ singularities to be located at $0, s_1, s_2, \dots, s_{2m-1}$ with $0 < s_1 < s_2 < \dots < s_{2m-1} < 2\pi R$. A necessary condition for the closure is

$$\sum_{i=1}^m s_{2i-1} - \sum_{i=1}^{m-1} s_{2i} = \pi R. \quad (49)$$

Indeed the total length of the region where k is positive must be equal to the total length where k is negative and must thus be equal to one half of the length $2\pi R$ of the circle.

We expect to come back to the problem of the warped Kaluza-Klein towers with singularities in a forthcoming paper. This change of metric has a direct impact on the establishment of the boundary conditions and the treatment of the Kaluza-Klein equations.

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Table 1: Two boundary conditions

Two Boundary Conditions		
Case	Boundary Conditions	Reduced Parameters
A1	$\phi(2\pi R) = e^{4\pi k R} (\alpha_1 \phi(0) + \alpha_2 \partial_s \phi(0))$ $\partial_s \phi(2\pi R) = e^{4\pi k R} \left(\frac{\alpha_1 \alpha_4 - 1}{\alpha_2} \phi(0) + \alpha_4 \partial_s \phi(0) \right)$	$\alpha_1 = \bar{\alpha}_1$, $\alpha_2 = \bar{\alpha}_2 / M_{Pl} \neq 0$ $\alpha_4 = \bar{\alpha}_4$
A2	$\phi(2\pi R) = e^{4\pi k R} \alpha_1 \phi(0)$ $\partial_s \phi(2\pi R) = e^{4\pi k R} \left(\alpha_3 \phi(0) + \frac{1}{\alpha_1} \partial_s \phi(0) \right)$	$\alpha_1 = \bar{\alpha}_1 \neq 0$ $\alpha_3 = \bar{\alpha}_3 M_{Pl}$
A3	$\partial_s \phi(0) = \rho_1 \phi(0)$ $\partial_s \phi(2\pi R) = \rho_2 \phi(2\pi R)$	$\rho_1 = \bar{\rho}_1 M_{Pl}$ $\rho_2 = \bar{\rho}_2 M_{Pl}$
A4	$\phi(0) = 0$ $\partial_s \phi(2\pi R) = \kappa \phi(2\pi R)$	$\kappa = \bar{\kappa} M_{Pl}$
A5	$\phi(2\pi R) = 0$ $\partial_s \phi(0) = \zeta \phi(0)$	$\zeta = \bar{\zeta} M_{Pl}$
A6	$\phi(0) = 0$ $\phi(2\pi R) = 0$	

Table 2: Three boundary conditions

Three Boundary Conditions		
Case	Boundary Conditions	Reduced Parameters
B1	$\phi(2\pi R) = e^{4\pi k R} \lambda_1 \phi(0)$ $\partial_s \phi(0) = \lambda_2 \phi(0)$ $\partial_s \phi(2\pi R) = e^{4\pi k R} \lambda_3 \phi(0)$	$\lambda_1 = \bar{\lambda}_1$ $\lambda_2 = \bar{\lambda}_2 M_{Pl}$ $\lambda_3 = \bar{\lambda}_3 M_{Pl}$
B2	$\phi(0) = 0$ $\phi(2\pi R) = e^{4\pi k R} \mu_1 \partial_s \phi(0)$ $\partial_s \phi(2\pi R) = e^{4\pi k R} \mu_2 \partial_s \phi(0)$	$\mu_1 = \bar{\mu}_1 / M_{Pl}$ $\mu_2 = \bar{\mu}_2$
B3	$\phi(0) = 0$ $\partial_s \phi(0) = 0$ $\partial_s \phi(2\pi R) = \nu \phi(2\pi R)$	$\nu = \bar{\nu} M_{Pl}$
B4	$\phi(0) = 0$ $\phi(2\pi R) = 0$ $\partial_s \phi(0) = 0$	

Table 3: Two boundary conditions : zero mass state constraints (\rightarrow : approximate relation resulting from $e^{4k\pi R}$ being very large)

Parameter constraints for zero mass states		
Case	Parameter constraint	σ_0, τ_0 relation
A1	$e^{8\pi k R} (4\alpha_1 \alpha_2 k - \alpha_1 \alpha_4 + 1) - 8 e^{4\pi k R} \alpha_2 k + (\alpha_1 \alpha_4 + 4\alpha_2 \alpha_4 k - 1) = 0$	$e^{4\pi k R} (e^{4\pi k R} - \alpha_1 - 4\alpha_2 k) \sigma_0 = (e^{4\pi k R} \alpha_1 - 1) \tau_0$
	$\rightarrow 4\alpha_1 \alpha_2 k - \alpha_1 \alpha_4 + 1 \approx 0$	$\rightarrow \sigma_0 \approx 0$
A2	$e^{8\pi k R} (4\alpha_1^2 k - \alpha_1 \alpha_3) - 8 e^{4\pi k R} \alpha_1 k + (\alpha_1 \alpha_3 + 4k) = 0$	$e^{4\pi k R} (e^{4\pi k R} - \alpha_1) \sigma_0 = (e^{4\pi k R} \alpha_1 - 1) \tau_0$
	$\rightarrow 4\alpha_1^2 k - \alpha_1 \alpha_3 \approx 0$	$\rightarrow \sigma_0 \approx 0$
A3	$e^{8\pi k R} \rho_1 (4k - \rho_2) + \rho_2 (\rho_1 - 4k) = 0$	$(4k - \rho_1) \sigma_0 = \rho_1 \tau_0$
	$\rightarrow \rho_1 (\rho_2 - 4k) \approx 0$	$\rightarrow (4k - \rho_1) \sigma_0 = \rho_1 \tau_0$
A4	$e^{8\pi k R} (4k - \kappa) + \kappa = 0$	$\sigma_0 = -\tau_0$
	$\rightarrow \kappa - 4k \approx 0$	$\rightarrow \sigma_0 = -\tau_0$
A5	$e^{8\pi k R} \zeta + (4k - \zeta) = 0$	$e^{8\pi k R} \sigma_0 = -\tau_0$
	$\rightarrow \zeta \approx 0$	$\rightarrow \sigma_0 \approx 0$
A6	No zero mass state	$\sigma_0 = \tau_0 = 0$

Table 4: Mass tower equations for two boundary conditions

Notations: $E = e^{2\pi k R}$, $F_0 = \frac{m}{k}$, $F_2 = e^{2\pi k R} \frac{m}{k}$		
Case	Mass equation	σ, τ relation
A1	$\begin{aligned} &\alpha_2 m E [(J_2(F_0)Y_1(F_2) - J_1(F_2)Y_2(F_0))\alpha_1 \\ &\quad + \alpha_2 m (J_1(F_0)Y_1(F_2) - J_1(F_2)Y_1(F_0))] \\ &+ [(\alpha_1\alpha_4 - 1)(J_2(F_2)Y_2(F_0) - J_2(F_0)Y_2(F_2)) \\ &\quad + \alpha_2\alpha_4 m (J_2(F_2)Y_1(F_0) - J_1(F_0)Y_2(F_2)) \\ &\quad - \alpha_2 \frac{4k}{\pi}] = 0 \end{aligned}$	$\begin{aligned} &(J_2(F_0)\alpha_1 + J_1(F_0)\alpha_2 m - J_2(F_2))\sigma \\ &\quad + (Y_2(F_0)\alpha_1 + Y_1(F_0)\alpha_2 m - Y_2(F_2))\tau = 0 \end{aligned}$
A2	$\begin{aligned} &m E \alpha_1^2 [J_2(F_0)Y_1(F_2) - J_1(F_2)Y_2(F_0)] \\ &+ [(J_2(F_2)Y_2(F_0) - J_2(F_0)Y_2(F_2))\alpha_1\alpha_3 \\ &\quad - \alpha_1 \frac{4k}{\pi} \\ &\quad + m (J_2(F_2)Y_1(F_0) - J_1(F_0)Y_2(F_2))] = 0 \end{aligned}$	$\begin{aligned} &(J_2(F_0)\alpha_1 - J_2(F_2))\sigma \\ &\quad + (Y_2(F_0)\alpha_1 - Y_2(F_2))\tau = 0 \end{aligned}$
A3	$\begin{aligned} &m E [\rho_1 (J_1(F_2)Y_2(F_0) - J_2(F_0)Y_1(F_2)) \\ &\quad + m (J_1(F_0)Y_1(F_2) - J_1(F_2)Y_1(F_0))] \\ &+ \rho_2 [\rho_1 (J_2(F_0)Y_2(F_2) - J_2(F_2)Y_2(F_0)) \\ &\quad + m (J_2(F_2)Y_1(F_0) - J_1(F_0)Y_2(F_2))] = 0 \end{aligned}$	$\begin{aligned} &(J_1(F_0)m - J_2(F_0)\rho_1)\sigma \\ &\quad + (Y_1(F_0)m - Y_2(F_0)\rho_1)\tau = 0 \end{aligned}$
A4	$\begin{aligned} &m E (J_1(F_2)Y_2(F_0) - J_2(F_0)Y_1(F_2)) \\ &\quad + \kappa (J_2(F_0)Y_2(F_2) - J_2(F_2)Y_2(F_0)) = 0 \end{aligned}$	$J_2(F_0)\sigma + Y_2(F_0)\tau = 0$
A5	$\begin{aligned} &\zeta (J_2(F_2)Y_2(F_0) - J_2(F_0)Y_2(F_2)) \\ &\quad + m (J_1(F_0)Y_2(F_2) - J_2(F_2)Y_1(F_0)) = 0 \end{aligned}$	$J_2(F_2)\sigma + Y_2(F_2)\tau = 0$
A6	$J_2(F_2)Y_2(F_0) - J_2(F_0)Y_2(F_2) = 0$	$J_2(F_0)\sigma + Y_2(F_0)\tau = 0$

Table 5: Tachyon equations for two boundary conditions

Notations: $E = e^{2\pi k R}$, $F_0 = \frac{h}{k}$, $F_2 = e^{2\pi k R} \frac{h}{k}$ ($m^2 = -h^2$)		
Case	Mass equation	σ, τ relation
A1	$\alpha_2 h E [(-I_2(F_0)K_1(F_2) - I_1(F_2)K_2(F_0))\alpha_1$ $+ \alpha_2 h (-I_1(F_0)K_1(F_2) + I_1(F_2)K_1(F_0))] + [(\alpha_1\alpha_4 - 1)(I_2(F_2)K_2(F_0) - I_2(F_0)K_2(F_2))$ $+ \alpha_2\alpha_4 h (-I_2(F_2)K_1(F_0) - I_1(F_0)K_2(F_2))$ $+ 2\alpha_2 k] = 0$	$(I_2(F_0)\alpha_1 + I_1(F_0)\alpha_2 h - I_2(F_2))\sigma$ $+ (K_2(F_0)\alpha_1 - K_1(F_0)\alpha_2 h - K_2(F_2))\tau = 0$
A2	$\alpha_1^2 h E [(-I_2(F_0)K_1(F_2) - I_1(F_2)K_2(F_0))$ $+ [(I_2(F_2)K_2(F_0) - I_2(F_0)K_2(F_2))\alpha_1\alpha_3$ $+ h (-I_2(F_2)K_1(F_0) - I_1(F_0)K_2(F_2))$ $+ 2\alpha_1 k] = 0$	$(I_2(F_0)\alpha_1 - I_2(F_2))\sigma$ $+ (K_2(F_0)\alpha_1 - K_2(F_2))\tau = 0$
A3	$h E [\rho_1 (I_1(F_2)K_2(F_0) + I_2(F_0)K_1(F_2))$ $+ h (-I_1(F_0)K_1(F_2) + I_1(F_2)K_1(F_0))] + \rho_2 [\rho_1 (I_2(F_0)K_2(F_2) - I_2(F_2)K_2(F_0))$ $- h (I_2(F_2)K_1(F_0) + I_1(F_0)K_2(F_2))] = 0$	$(I_1(F_0)h - I_2(F_0)\rho_1)\sigma$ $- (K_1(F_0)h + K_2(F_0)\rho_1)\tau = 0$
A4	$h E (I_1(F_2)K_2(F_0) + I_2(F_0)K_1(F_2))$ $+ \kappa (I_2(F_0)K_2(F_2) - I_2(F_2)K_2(F_0)) = 0$	$I_2(F_0)\sigma + K_2(F_0)\tau = 0$
A5	$\zeta (I_2(F_2)K_2(F_0) - I_2(F_0)K_2(F_2))$ $+ h (I_1(F_0)K_2(F_2) + I_2(F_2)K_1(F_0)) = 0$	$I_2(F_2)\sigma + K_2(F_2)\tau = 0$
A6	$I_2(F_2)K_2(F_0) - I_2(F_0)K_2(F_2) = 0$	$I_2(F_0)\sigma + K_2(F_0)\tau = 0$

Table 6:

Towers of mass eigenvalues for the Case A4, masses are in TeV, $\bar{k}\bar{R} = 6.3$, $\bar{k} = 1$											
\bar{k}	h	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
- 100.0		0.4	0.655	0.9048	1.152	1.398	1.644	1.889	2.134	2.379	2.624
- 12.0		0.3769	0.62	0.8599	1.1	1.34	1.582	1.825	2.068	2.312	2.557
- 8.0		0.3672	0.6072	0.846	1.086	1.328	1.571	1.814	2.058	2.303	2.548
- 4.0		0.349	0.5866	0.8265	1.069	1.312	1.557	1.802	2.047	2.293	2.539
0.0		0.301	0.5512	0.7993	1.047	1.294	1.542	1.789	2.036	2.283	2.529
3.9		0.0605	0.5025	0.768	1.024	1.276	1.526	1.775	2.024	2.272	2.52
3.99		0.0192	0.5014	0.766	1.023	1.275	1.525	1.774	2.023	2.272	2.52
3.999		0.00609	0.5016	0.766	1.023	1.275	1.525	1.775	2.023	2.272	2.52
4.0		0	0.5013	0.766	1.023	1.275	1.525	1.775	2.023	2.272	2.52
4.001	0.00609		0.5016	0.766	1.023	1.275	1.525	1.775	2.023	2.272	2.52
4.01	0.01926		0.5016	0.766	1.023	1.275	1.525	1.775	2.023	2.272	2.52
4.1	0.0612		0.501	0.766	1.022	1.275	1.525	1.774	2.023	2.271	2.519
8.0	0.4826		0.4624	0.7380	1.0	1.256	1.51	1.761	2.011	2.261	2.51
12.0	0.8091		0.4428	0.718	0.9819	1.241	1.496	1.749	2.0	2.251	2.501
16.0	1.128		0.4327	0.7058	0.9689	1.228	1.484	1.737	1.99	2.241	2.492
20.0	1.445		0.426	0.6969	0.959	1.218	1.474	1.728	1.981	2.233	2.484
100.0	7.737		0.408	0.6689	0.9227	1.175	1.426	1.676	1.926	2.176	2.426

Table 7:

Towers of mass eigenvalues for the Case A1, masses are in TeV, $\bar{k}\bar{R} = 6.3$, $\bar{k} = 1$, $\bar{\alpha}_1 = 0.7$, $\bar{\alpha}_4 = 6.6286$											
$\bar{\alpha}_2$	h	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
0.	∞		0.4035	0.6613	0.9129	1.162	1.411	1.659	1.907	2.154	2.402
0.02	20.31		0.4051	0.6639	0.9165	1.167	1.416	1.665	1.914	2.163	2.411
0.2	1.918		0.4204	0.6884	0.949	1.206	1.462	1.716	1.969	2.222	2.473
1.	0.2269		0.4873	0.7575	1.015	1.269	1.521	1.771	2.019	2.268	2.516
1.29	0.0340		0.5012	0.766	1.022	1.274	1.525	1.774	2.023	2.271	2.519
1.299	0.0107		0.5013	0.7669	1.022	1.275	1.525	1.774	2.023	2.271	2.519
1.3		0	0.5013	0.766	1.023	1.275	1.525	1.774	2.023	2.271	2.519
1.301		0.01064	0.5013	0.766	1.023	1.275	1.525	1.774	2.023	2.271	2.519
1.31		0.03359	0.5017	0.766	1.023	1.275	1.525	1.774	2.023	2.271	2.519
1.6		0.159	0.5112	0.773	1.027	1.278	1.528	1.777	2.025	2.273	2.521
2.6		0.24	0.5269	0.7833	1.035	1.284	1.533	1.781	2.029	2.277	2.524
40.0		0.2984	0.5497	0.7983	1.046	1.293	1.541	1.788	2.035	2.282	2.529
100.0		0.3	0.5511	0.7989	1.046	1.294	1.541	1.788	2.035	2.282	2.529
$\pm\infty$		0.3010	0.5512	0.7993	1.047	1.294	1.541	1.788	2.035	2.282	2.529
- 100.0		0.3021	0.5518	0.7997	1.047	1.294	1.541	1.788	2.035	2.282	2.529
- 40.0		0.3037	0.5527	0.8011	1.048	1.295	1.542	1.789	2.035	2.282	2.529
- 2.0		0.3375	0.5763	0.8178	1.061	1306.	1.551	1.797	2.043	2.289	2.535
- 0.001		0.4034	0.6612	0.9127	1.162	1.411	1.659	1.906	2.154	2.402	2.648

Table 8:

Towers of mass eigenvalues for the Case A3, masses are in TeV, $\bar{k}\bar{R} = 6.3$, $\bar{k} = 1$											
$\bar{\rho}_2$	h	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
- 4000.0		0.4034	0.6611	0.9127	1.162	1.411	1.659	1.906	2.154	2.401	2.648
- 400.0		0.4025	0.6596	0.9106	1.160	1.407	1.655	1.902	2.149	2.396	2.642
- 40.0		0.3940	0.6459	0.8919	1.136	1.380	1.623	1.866	2.109	2.352	2.596
- 4.0		0.3487	0.5866	0.8265	1.069	1.312	1.557	1.802	2.047	2.293	2.539
0.0		0.3010	0.5512	0.7993	1.047	1.294	1.541	1.788	2.035	2.282	2.529
3.0		0.1807	0.5139	0.7750	1.029	1.279	1.529	1.778	2.026	2.274	2.522
3.9		0.0605	0.5025	0.766	1.023	1.275	1.525	1.775	2.023	2.271	2.519
3.99		0.0192	0.5014	0.766	1.023	1.275	1.525	1.774	2.023	2.271	2.519
4.0		0.	0.5012	0.766	1.023	1.275	1.525	1.774	2.023	2.271	2.519
4.01	0.0193		0.5011	0.766	1.023	1.275	1.525	1.774	2.023	2.271	2.519
4.1	0.0612		0.5000	0.766	1.022	1.274	1.524	1.774	2.023	2.271	2.519
5.0	0.2046		0.4895	0.7590	1.017	1.270	1.521	1.771	2.020	2.268	2.517
8.0	0.4826		0.4624	0.7380	1.0	1.256	1.509	1.761	2.011	2.260	2.509
12.0	0.8092		0.4425	0.7180	0.9818	1.240	1.495	1.748	2.0	2.250	2.500
40.0	3.021		0.4142	0.6786	0.9365	1.192	1.446	1.699	1.952	2.204	2.455
400.0	31.31		0.4045	0.6629	0.9152	1.165	1.415	1.663	1.912	2.160	2.408
4000.0	314.1		0.4036	0.6614	0.9131	1.163	1.411	1.659	1.907	2.155	2.402

Table 9: The relative probability densities of the ten first mass eigenvalues $m_{i,0}$ (in TeV) for given $x = s/(2\pi R)$ in the tower corresponding to the Case A1 with $kR = 6.3$, $\bar{k} = 1$, $\bar{\alpha}_1 = 0.7$, $\bar{\alpha}_2 = 1.3$, $\bar{\alpha}_4 = 6.6286$. The highest probability among these ten masses is normalized to exactly 1 and labeled as such. Note that the physical masses depend on s_0 , hence on x , and are given in terms of the mass eigenvalues by $m_{i,s_0} = m_{i,0}e^{2\pi x k R}$ in agreement with Eq.(34).

Mass eigenvalues	$m_{1,0}$	$m_{2,0}$	$m_{3,0}$	$m_{4,0}$	$m_{5,0}$	$m_{6,0}$	$m_{7,0}$	$m_{8,0}$	$m_{9,0}$	$m_{10,0}$
	0	0.5013	0.766	1.023	1.275	1.525	1.774	2.023	2.271	2.519
x	Relative Probabilities									
0	1	0.27	0.19	0.14	0.11	0.096	0.083	0.073	0.065	0.059
0.4	1	0.27	0.19	0.14	0.11	0.096	0.083	0.073	0.065	0.059
0.45	1	0.27	0.19	0.14	0.12	0.084	0.10	0.055	0.090	0.034
0.46	1	0.26	0.21	0.11	0.17	0.044	0.18	0.0085	0.23	0.0011
0.466	1	0.24	0.25	0.067	0.27	0.0024	0.41	0.035	0.69	0.22
0.467	1	0.24	0.26	0.057	0.30	0.000021	0.49	0.071	0.86	0.35
0.4676	1	0.24	0.27	0.051	0.33	0.00066	0.55	0.10	0.99	0.45
0.4676334	1	0.24	0.27	0.050	0.33	0.00075	0.55	0.10	1.0	0.46
0.4676335	1.0	0.24	0.27	0.050	0.33	0.00075	0.55	0.10	1	0.46
0.4677	0.98	0.23	0.26	0.049	0.33	0.00095	0.55	0.11	1	0.46
0.468	0.92	0.21	0.25	0.042	0.32	0.0021	0.54	0.12	1	0.49
0.47	0.56	0.12	0.17	0.013	0.26	0.018	0.50	0.19	1	0.67
0.475	0.14	0.022	0.068	0.0017	0.16	0.081	0.42	0.36	0.96	1
0.48	0.026	0.0017	0.027	0.012	0.10	0.11	0.31	0.39	0.78	1
0.5	0.00012	0.0013	0.0092	0.027	0.067	0.13	0.25	0.41	0.66	1
0.9	0.000022	0.0017	0.0092	0.029	0.069	0.14	0.25	0.42	0.67	1
0.92	0.000028	0.0022	0.011	0.035	0.083	0.16	0.29	0.46	0.70	1
0.94	0.00011	0.0081	0.040	0.11	0.23	0.40	0.59	0.78	0.93	1
0.95	0.00052	0.036	0.16	0.40	0.69	0.93	1	0.86	0.56	0.24
0.97	0.012	0.51	1	0.58	0.017	0.21	0.40	0.095	0.044	0.26
0.99	0.71	0.85	1	0.084	0.24	0.49	0.065	0.13	0.33	0.060
0.995	1	0.0075	0.081	0.25	0.31	0.21	0.064	0.000017	0.048	0.13
0.999	1	0.26	0.16	0.11	0.073	0.049	0.031	0.019	0.010	0.0044
1	1	0.27	0.19	0.14	0.11	0.096	0.083	0.073	0.065	0.059

Table 10: The relative probability densities of the ten first physical masses m_i (in TeV) for given $x_0 = s_0/(2\pi R)$ in the tower corresponding to the Case A1, $\bar{k} = 1$, $\bar{\alpha}_1 = 0.7$, $\bar{\alpha}_2 = 1.3$, $\bar{\alpha}_4 = 6.6286$. The highest probability among these ten masses is normalized to 1 and labeled as such. When x_0 is increased, \bar{R} is adjusted according to (34), keeping $\bar{k} = 1$, so as to retain the physical masses unchanged.

Physical masses		m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
		0	0.5013	0.766	1.023	1.275	1.525	1.774	2.023	2.271	2.519
x_0	\bar{R}	Relative Probabilities									
0	6.3	1	0.28	0.19	0.14	0.11	0.096	0.083	0.073	0.065	0.059
0.44	11.3	1	0.27	0.18	0.14	0.11	0.095	0.082	0.072	0.064	0.058
0.475	12.0	1	0.27	0.19	0.13	0.13	0.077	0.11	0.046	0.1	0.025
0.4793	12.1	1	0.26	0.21	0.11	0.17	0.041	0.18	0.0068	0.24	0.0022
0.4815	12.15	1	0.25	0.23	0.081	0.22	0.012	0.3	0.0062	0.47	0.088
0.4828	12.18	1	0.24	0.25	0.059	0.29	0.00041	0.45	0.056	0.79	0.29
0.4832	12.19	1	0.23	0.26	0.051	0.32	0.00033	0.52	0.093	0.94	0.42
0.4834	12.195	0.97	0.22	0.26	0.045	0.32	0.0015	0.55	0.11	1	0.48
0.4836	12.2	0.88	0.2	0.24	0.037	0.31	0.0033	0.54	0.13	1	0.51
0.4857	12.25	0.32	0.062	0.11	0.0012	0.21	0.046	0.47	0.28	1	0.85
0.49	12.3	0.095	0.013	0.053	0.0041	0.14	0.09	0.38	0.37	0.9	1
0.61	16.3	0.000021	0.0017	0.0088	0.028	0.066	0.14	0.25	0.42	0.66	1
0.76	26.3	0.000022	0.0017	0.0089	0.028	0.067	0.14	0.25	0.42	0.69	1
0.89	56.3	0.000021	0.0016	0.0081	0.025	0.07	0.12	0.25	0.4	0.6	1

Figure 1: Case A1 : The logarithm of the field probability density as a function of $x = s/(2\pi R)$ for $\bar{k} = 1$, $kR = 6.3$, $\bar{\alpha}_1 = 0.7$, $\bar{\alpha}_2 = 1.3$, $\bar{\alpha}_4 = 6.6286$ and $m_1 = 0$

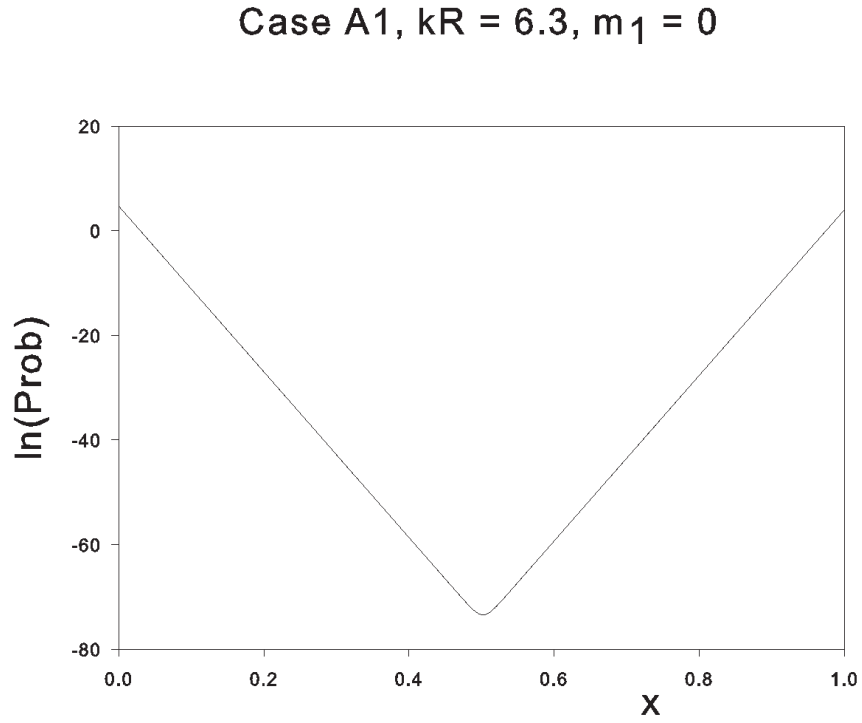


Figure 2: Case A1 : The logarithm of the field probability density as a function of $x = s/(2\pi R)$ for $\bar{k} = 1$, $kR = 6.3$, $\bar{\alpha}_1 = 0.7$, $\bar{\alpha}_2 = 1.3$, $\bar{\alpha}_4 = 6.6286$ and $m_2 = 0.501\text{TeV}$

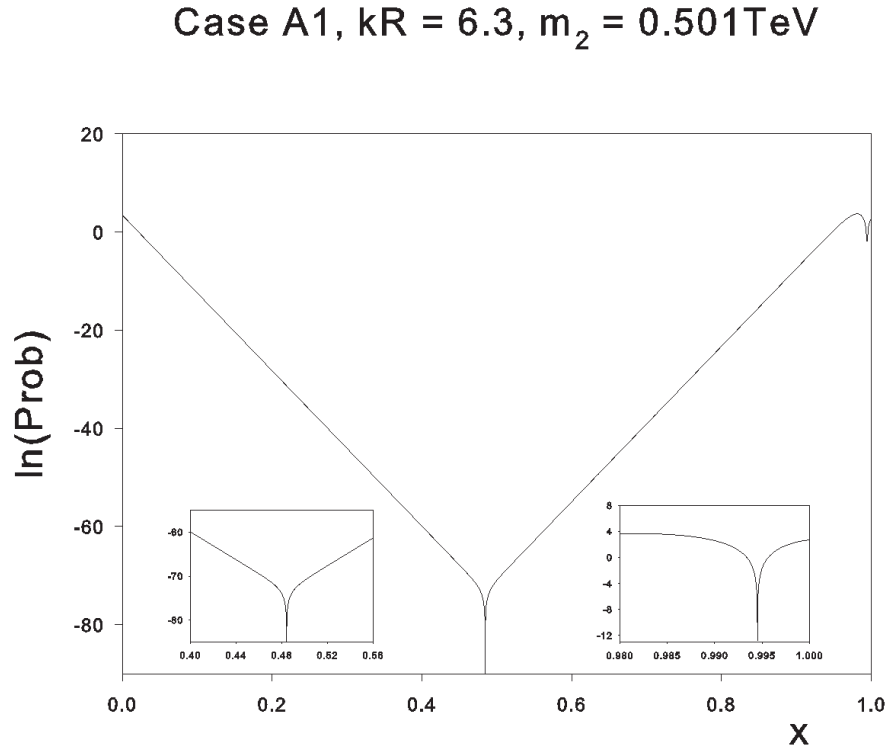


Figure 3: Case A1 : The logarithm of the field probability density as a function of $x = s/(2\pi R)$ for $\bar{k} = 1$, $kR = 6.3$, $\bar{\alpha}_1 = 0.7$, $\bar{\alpha}_2 = 1.3$, $\bar{\alpha}_4 = 6.6286$ and $m_5 = 1.275\text{TeV}$

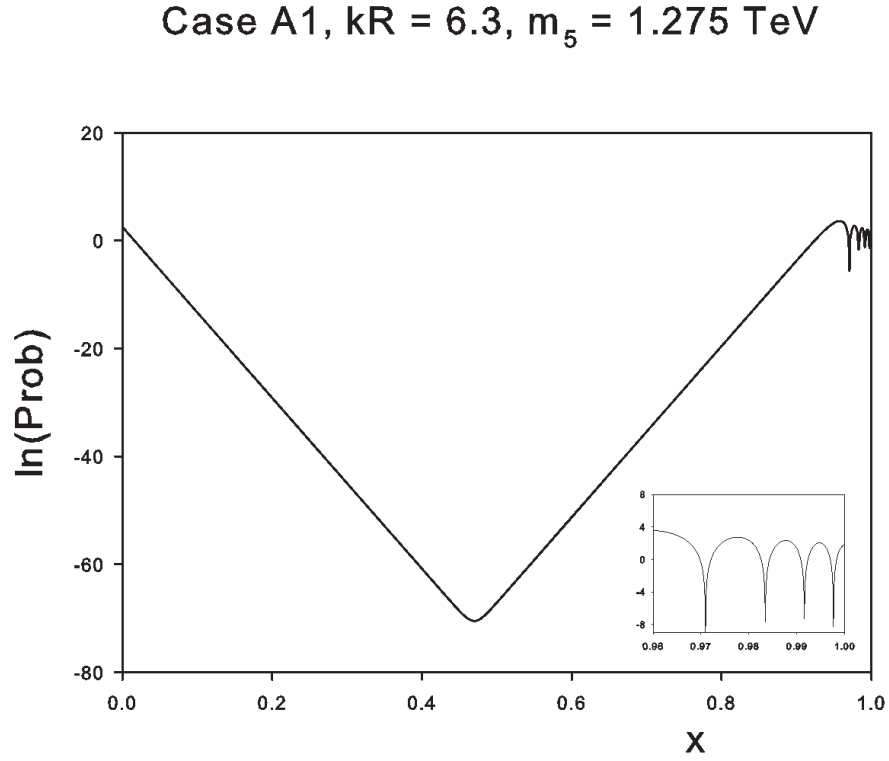


Figure 4: Case A4 : The logarithm of the field probability density as a function of $x = s/(2\pi R)$ for $\bar{k} = 1$, $kR = 6.3$, $\bar{\kappa} = 4$ and $m_1 = 0$

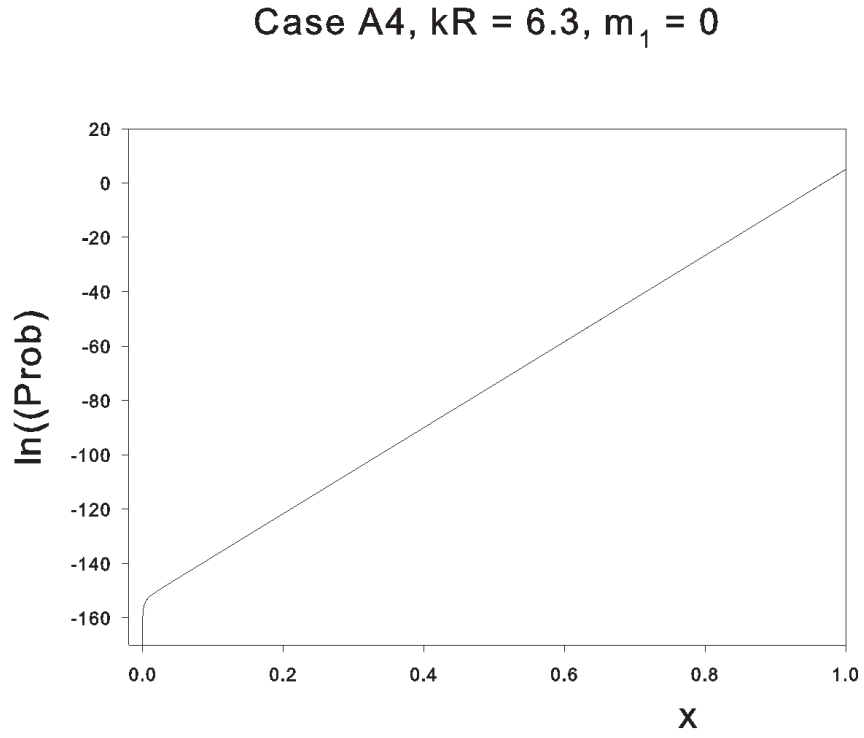


Figure 5: Case A4 : The logarithm of the filed probability density as a function of $x = s/(2\pi R)$ for $\bar{k} = 1$, $kR = 6.3$, $\bar{\kappa} = 4$ and $m_3 = 0.766$ TeV

